Lecture 26

Measure and Integration

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$E \subseteq X$, $F \subseteq Y$, $E \subseteq F$

$x \in X$

$E_x := \{ y \in Y \mid (x,y) \in E \}$

Note $y \in Y$, $(x,y) \in E \subseteq F$

$\Rightarrow$ $y \in F_x$ also.

$\Rightarrow$ $E_x \subseteq F_x$

Similarly $E^y \subseteq F^y$.

$$E, F \subseteq X \times Y$$

$$x \in X$$

$$(E \setminus F)_x = \{ y \in Y \mid (x,y) \in E \setminus F \}$$

$$y \in (E \setminus F)_x \Leftrightarrow (x,y) \in E \setminus F$$

$$\Leftrightarrow (x,y) \in E, (x,y) \notin F$$

$$\Rightarrow y \in E_x, \ y \notin F_x$$

$$\Leftrightarrow y \in E_x \setminus F_x$$

$$(E \setminus F)_x \subseteq E_x \setminus F_x$$

$$\supseteq$$

$E_i \subseteq X \times Y,\ i \in I,\ x \in X$

$$\boxed{\left(\bigcap_{i \in I} E_i\right)_x = \bigcap_{i \in I} (E_i)_x}$$

$$y \in \left(\bigcap_{i \in I} E_i\right)_x \iff (x,y) \in \bigcap_{i \in I} E_i$$

$$\iff (x,y) \in E_i \ \forall i \in I$$

$$\iff y \in (E_i)_x \ \forall i \in I$$

$$\iff y \in \bigcap_{i \in I} (E_i)_x$$

$E_i \subseteq X \times Y,\ i \in I,\ x \in X$

$$\left(\bigcup_{i\in I} E_i\right)_x = \bigcup_{i\in I} (E_i)_x \quad \checkmark$$

$$y \in \left(\bigcup_{i\in I} E_i\right)_x \iff (x,y) \in \bigcup_{i\in I} E_i$$

$$\iff (x,y) \in E_i \text{ for some } i \in I$$

$$\iff y \in (E_i)_x \text{ for some } i$$

$$\iff y \in \bigcup_{i\in I} (E_i)_x.$$

$E \in \mathcal{A} \otimes \mathcal{B}$.

$$\mathcal{S} = \{ E \in \mathcal{A} \otimes \mathcal{B} \mid E_x \in \mathcal{B}, E^y \in \mathcal{A} \}$$

To show $\mathcal{S} = \mathcal{A} \otimes \mathcal{B}$

(i) $\underline{\mathcal{R} \subseteq \mathcal{S}}$: Let $A \in \mathcal{A}$, $B \in \mathcal{B}$

$E = A \times B$

$$(E)_x = \{ y \in Y \mid (x, y) \in A \times B \}$$

$$= \begin{cases} B & \forall x \in A \\ \phi & x \notin A \end{cases}$$

$\in \mathcal{B}$.

(ii) $\mathcal{S}$ is a σ-algebra

$$(\varphi)_x = (\varphi)_y = \varphi \in \cancel{\mathcal{A}\times\mathcal{B}}\ \mathcal{R} \subseteq \mathcal{S}$$

$$(X \times Y) \in \mathcal{R} \subseteq \cancel{\mathcal{A}\otimes\mathcal{B}}\ \mathcal{S}.$$

$$\begin{aligned}
E \in \mathcal{S} &\Longrightarrow E_x \in \mathcal{B},\ E_y \in \mathcal{A} \\
&\Longrightarrow (E_x)^c \in \mathcal{B},\ (E_y)^c \in \mathcal{A} \\
&\qquad\ \ \parallel \qquad\qquad\quad \parallel \\
&\qquad (E^c)_x \in \mathcal{B},\ (E^c)_y \in \mathcal{A} \\
&\Longrightarrow E^c \in \mathcal{S}
\end{aligned}$$

Let $E_i \in \Sigma$, $i \geq 1$

$$\forall i,\ E_i \in \Sigma \Rightarrow (E_i)_x \in \mathcal{B},\ (E_i)^y \in \mathcal{A}$$

$$\Rightarrow \bigcup_{i=1}^{\infty} (E_i)_x \in \mathcal{B},\ \left(\bigcup_{i=1}^{\infty} E_i^y\right) \in \mathcal{A}$$

$$\Rightarrow \left(\bigcup_{i=1}^{\infty} E_i\right)_x \in \mathcal{B},\ \left(\bigcup_{i=1}^{\infty} E_i\right)^y \in \mathcal{A}$$

$$\Rightarrow \bigcup_{i=1}^{\infty} E_i \in \Sigma.$$

Hence Σ is a σ-algebra, &

$\mathcal{R} \subseteq \Sigma \Rightarrow \mathcal{A} \otimes \subseteq \Sigma \subseteq \mathcal{A} \otimes \mathcal{B}$

$$\mathcal{P} = \left\{ E \in \mathcal{A} \otimes \mathcal{B} \;\middle|\; \begin{array}{l} x \longmapsto \nu(E_x) \\ y \longmapsto \mu(E_y) \end{array} \right\} \text{mble}$$

$$\int_X \nu(E_x)\,d\mu = \int_X \mu(E^y)\,d\nu = (\mu\times\nu)(E) \Big\}$$

(1) $\underline{\mathcal{R} \subseteq \mathcal{P}}$

$E = A \times B,\quad E_x = \begin{cases} \phi & \text{if } x \notin A \\ B & \text{if } x \in A \end{cases}$

$\nu(E_x) = \nu(B)\chi_A(x)$ ——— $\circledast$

$\Rightarrow \quad x \longmapsto \nu(E_x)$ is $\mathcal{A}$-mble

$$\cancel{E_y = \begin{cases} \phi & \forall\, x \notin B \\ A & \end{cases}}$$

$$E^y = \begin{cases} \phi & \forall\, y \notin B \\ A & \forall\, y \in B \end{cases}$$

$$\mu(E^y) = \mu(A)\,\chi_B(y) \quad \text{———} \quad (**)$$

$\Rightarrow \quad y \longmapsto \mu(E^y)$ is $\mathcal{B}$-msbl.

From ⊛

$$\int \nu(E_x)\,d\mu(x) = \nu(B)\mu(A)$$
$$= (\mu\times\nu)(A\times B)$$

From ⊛⊛

$$\int \mu(E^y)\,d\nu(y) = \mu(A)\nu(B)$$
$$= (\mu\times\nu)(A\times B)$$

Hence $\mathcal{R} \subseteq \mathcal{P}$.

$\mathscr{S}$ is closed under finite disjoint unions:

$$E, F \in \mathscr{S},\ E \cap F = \emptyset$$

$$\overset{?}{\Longrightarrow} \quad E \cup F \in \mathscr{S}\,?$$

$$(E \cup F)_x = (E_x) \cup (F_x)$$

$$\nu\left[(E \cup F)_x\right] = \nu(E_x \cup F_x)$$

$$= \underline{\nu(E_x)} + \underline{\nu(F_x)} \qquad (!)$$

$$\Longrightarrow \quad x \longmapsto \nu((E \cup F)_x) \text{ is } \mathcal{A}\text{-msbl.}$$

|||ly $$y \longmapsto \mu((E \cup F)^y) \text{ is } \mathcal{B}\text{-msbl.}$$

$$\int_X \nu((E \cup F)_x)\, d\mu(x)$$

$$= \int_X [\nu(E_x) + \nu(F_x)]\, d\mu(x)$$

$$= \int_X \nu(E_x)\, d\mu(x) + \int_X \nu(F_x)\, d\mu(x)$$

$$= (\mu \times \nu)(E) + (\mu \times \nu)(F)$$

$$= (\mu \times \nu)(E \cup F)$$

Similarly

$$\int_Y \mu((E \cup F)^y)\, d\nu(y)$$

$$= \int_Y \left[\mu(E^y) + \mu(F^y)\right] d\nu(y)$$

$$= (\mu \times \nu)(E) + (\mu \times \nu)(F)$$

$$= (\mu \times \nu)(E \cup F).$$